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MATHEMATICAL TECHNIQUES FOR COST-EFFECTIVENESS OPTIMIZATION. (U)

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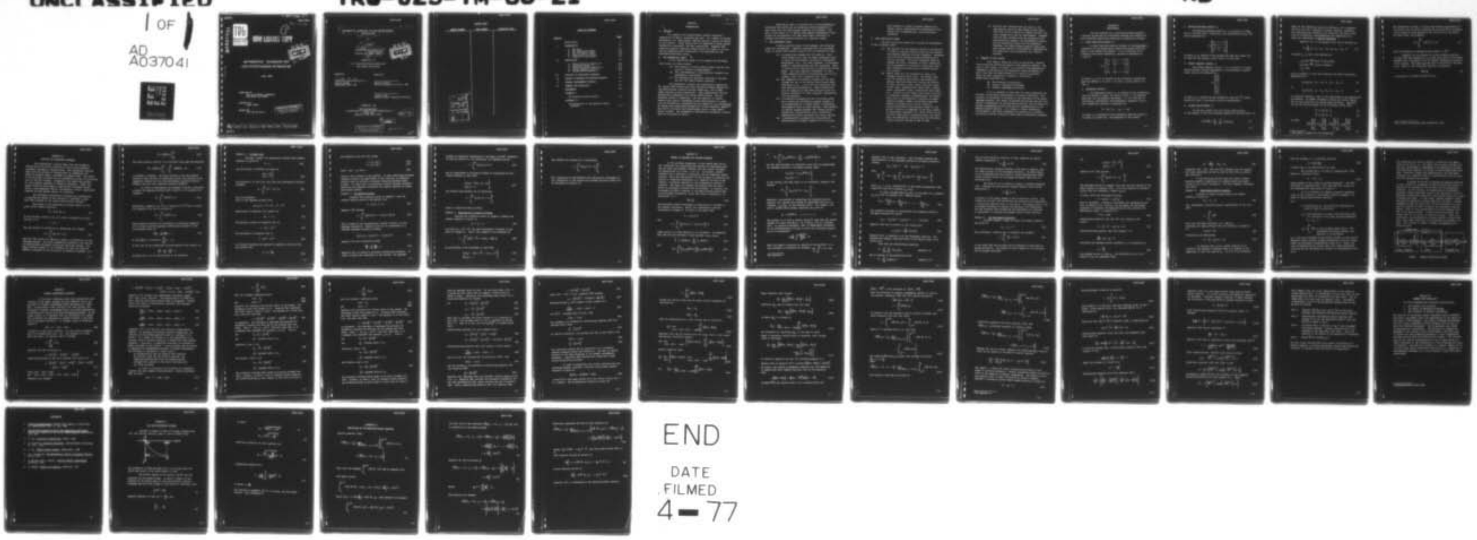

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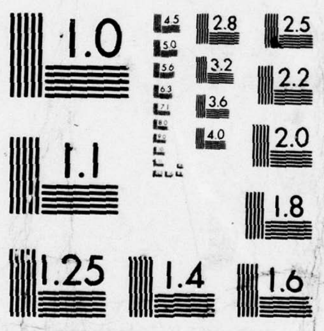

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TABLE OF CONTENTS

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SECTION I

INTRODUCTION

A. THE GOAL

A complete cost-effectiveness analysis requires a combination of theory, procedure and data. As a result of studies performed on all three phases, TRG has planned a series of reports on this subject. These reports will be organized in three groups: theoretical, procedural, and data acquisition. Together they comprise a complete analytical method which will apply to the system and subsystem trade-off analyses to be performed for the Conformal/Planar Sonar Array.

B. THE THEORETICAL PHASE

In the theoretical phase of the program the following three areas are being studied.

(a) The concepts of system effectiveness from the view point of "mission accomplishment".

(b) The transformation of philosophical concepts into mathematical expressions.

(c) The mathematical techniques required to optimize the cost and effectiveness equations.

The fundamental concepts and mathematical expressions for system effectiveness have been defined by WSEIAC (Weapon System Effectiveness Industry Advisory Committee), and by NASL (Naval Applied Science Laboratory). They independently derived mathematical expressions for system effectiveness. The two equations differ in appearance but are identical in theory because both schools considered system effectiveness as a function of availability, dependability and capability (or performance). By making some simplifying assumptions in their equation, WSEIAC[1] was able to calculate the effectiveness value of some simple weapon systems. The mathematical optimization techniques, however, were not discussed in detail.

Essentially, what is required now is some mathematical techniques with which one can manipulate system parameters in such a way that the effectiveness equation is optimized. The goal of this report is to develop some general mathematical techniques for optimization, and then apply these techniques to optimize the effectiveness equation of a specific system.

C. THE PROCEDURAL PHASE

In the procedural phase of the program, the following steps are contemplated. (Reports concerned with all procedural steps in system effectiveness analysis will be submitted to NEL.)

(a) A system block diagram will be drawn up. This block diagram will show all major components (such as front end electronics, TR switch, transducers, beamformers, etc.) For each component, a reasonable number of implementations will be considered. Each of these will be characterized by significant failure modes. For each component considered, at least three different levels of availability will be considered, each with a different cost level.

(b) The comparison of different systems will be made through the use of the system effectiveness model developed in (a). A cost will be associated with each selected configuration. These combinations will be examples for CNO, DOD evaluation and convenient evaluation of other component combinations will be possible.

(c) A reliability program plan will be written which will contain the reliability program elements to be accomplished by a company which is performing developmental work on sonar systems. Some examples of typical program elements are: design reviews, predictions, qualification tests, reliability demonstration tests and worst-case error analysis.

(d) A maintainability program plan containing the maintainability program elements will be written.

Some examples of typical program elements are: design review, maintainability demonstration test, spares test recommendations, plan maintenance schedules.

D. DATA ACQUISITION PHASE

The data acquisition phase will include the performance of the following tasks:

(a) The analysis of the predicted values of reliability in terms of constant failure rates for the various environments expected for that component. Estimated costs for each proposed design alternative will also be included. For those components where no designs yet exist, estimates of their reliability will be generated on the basis of the best available information using various standard failure rate sources.

(b) Repair rates will be estimated in terms of an assumed constant repair rate. The repair rates will be estimated from repair time probability density functions which will include the factors of fault detection times, fault isolation times, access times, administrative delay times, removal times, replacement times, verification test times, etc. Instead of this repair rate being a function of different component configurations, the repair rates will be considered as a function of spare/logistic schemes, number of available repairmen, degree of automaticity of fault isolation, continuous/discrete time testing, number of test points, location of component on the ship, degree of restoration required, etc. Fault detection and location times will be correlated with, and taken from, the FISC (Fault Isolation and Self-Checking) effort presently pursued by TRG for NEL.

(c) With the data obtained from (a) and (b), the availability vector of the system will be found. Then this can be studied as a Markov process, but other schemes will also be investigated. Together with the data obtained from the reliability and maintainability program, the dependability matrix and the capability vector will be formulated. Therefore, the effectiveness equation shall have been defined (based on availability, dependability and capability) and optimization will then be possible.

E. SUMMARY OF THIS REPORT

This report, being the first in the series, concerns itself with the theoretical phase of the complete analysis. The goal of this report, to repeat, is to develop some mathematical techniques with which one can manipulate system parameters in such a way that the effectiveness of a system is optimized. There are many techniques to optimize a functional. Among those techniques are:

(a) The calculus of Variations.
(b) Theory of Maximum and Minimum.
(c) Dynamic Programming Principle.

It is of interest to note that all three techniques are related to well-known formulations in classical mechanics. The calculus of variations is utilized in the derivation of the Euler-Lagrange equation, the theory of maximum and minimum resulted from special consideration of the Hamilton principle, and the theory of dynamic programming evolved from special considerations of the Hamilton-Jacobi equation. As a result of these interesting relationships, the three techniques are selected to illustrate the nature, and the solution of some simple optimization problems.

SECTION II
DEFINITIONS

For the purposes of a cost-effectiveness analysis, a typical system may be investigated through the analysis of four essential quantitites which may be defined as follows:

A. STATE TRANSITION MATRIX; R

A state transition matrix consists of distinguishable conditions of the system which result from events occurring prior to and during the mission. It is a function of reliability and repair capability, and may be expressed as a matrix:

$$R = \begin{bmatrix} r_{11} & r_{12} \cdots\cdots & r_{1n} \\ r_{21} & r_{22} \cdots\cdots & r_{2n} \\ \vdots & \vdots & \vdots \\ r_{n1} & r_{n2} \cdots\cdots & r_{nn} \end{bmatrix}$$

In which r_{ij} (o,t) is defined as the transition probability that the system is in state i at time t, given state ij at time o.

B. READINESS VECTOR; V

The Readiness Vector, V, is defined as the probability that the system is ready, and that at time, t, all required functions are successfully performed. V, is a function of reliability, maintenance, and maintenance diagnostic and repair capability; and may be expressed by:

$$V = (v_1, v_2, \ldots v_i, \ldots v_n)$$

In which v_i is defined as the probability that the system is in state i at time zero, the beginning of the mission.

C. MISSION READINESS MATRIX; W

The Mission Readiness Matrix W, is a function of diagnosis of system state, flexibility, and operational policy; and may be expressed by:

$$W = \begin{bmatrix} w_1 & o & o \ldots\ldots o \\ o & w_2 & o \ldots\ldots o \\ o & o & w_3 \ldots\ldots o \\ \vdots & \vdots & \\ o & o & o \ldots\ldots w_n \end{bmatrix}$$

In which w_i is defined as the probability that the system will be used for the mission, given state i at time zero.

D. DESIGN ADEQUACY VECTOR; D

The Design Adequacy Vector, D, is a function of design specifications, mission requirements, and performance capabilities; and may be expressed by:

$$D = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_i \\ \vdots \\ d_n \end{bmatrix}$$

In which d_i is defined as the probability that the i^{th} system state will lead to successful mission accomplishment.

E. SYSTEM EFFECTIVENESS, E

It has been shown[2] that the system effectiveness is the product of the four essential quantities defined above, or:

$$E \triangleq VRWD = \sum_{j=1}^{n} \sum_{i=1}^{n} v_i w_i r_{ij} d_j \qquad (1)$$

Based on the definitions given for VRW and D, it is apparent the E is not only a function of money, material, personnel and time which are referred to as resources, but also a function of reliability, maintainability and capability etc., which are identified as control variables.

Equation (1), in its general form, may be written as:

$$E = \sum_{i=1}^{n} f_i \ (x_1, x_2 \ldots . x_n, m_1, m_2 \ldots . m_n, t) \qquad (2)$$

In which x_i, m_i and t are defined as: *

$x_i \triangleq$ the $i^{\underline{th}}$ state of the system

$m_i \triangleq$ the $i^{\underline{th}}$ control variables

$t \triangleq$ time

m_i is a function of both the resources and their derivatives. Mathematically:

$$m_i \triangleq g_i(x_i, x_2 \ldots . x_n, \dot{x}_1, \dot{x}_2 \ldots . \dot{x}_n, t) \qquad (3)$$

or

$$\dot{x}_i \triangleq h_i(x_i, x_2 \ldots . x_n, m_1, m_2 \ldots . m_n, t) \qquad (4)$$

In classical mechanics, the x's are identified as the generalized coordinates. In modern information theory and control theory, the x's are referred to as system state variables. Equations (2) and (4) may be written in the following vector forms:

$$\overline{E} = F \ (\overline{x}, \overline{m}, t) \qquad (5)$$

$$\overline{\dot{X}} = H \ (\overline{x}, \overline{m}, t) \qquad (6)$$

In which

$$\overline{E} \triangleq \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}; \ \overline{x} \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}; \ \overline{\dot{x}} \triangleq \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}; \ \overline{m} \triangleq \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix}$$

* The symbol $\triangleq$ stands for "by definition"

The fundamental concept of system effectiveness optimization is to select a set of control variables $\overline{M}$, such that $\overline{E}$ is optimized over a period of time defined as the life cycle of the system. In other words, the integral:

$$\overline{E} = \int_{t_o}^{t_f} F(\overline{X}, \overline{M}, t)\, dt \qquad (7)$$

must be maximized or minimized with respect to $\overline{M}$.*

Equation (7) is referred to as the criterion function in which the time interval $t_f - t_o$ is the life cycle of the system. Equation (6) is referred to as the dynamic differential equation of the system. One other equation must be included in the analysis. That is, the constraint equation on resources:

$$|\overline{M}| \leq \overline{M}_o \qquad (8)$$

in which $\overline{M}_o$ is a finite constant vector.

*All vectors are written with superscript bars.

SECTION III

CALCULUS OF VARIATIONS APPROACH

The differential calculus deals with the problem of finding points at which a function of one or more variables possesses a maximum or minimum value. The variational calculus deals with the problem of finding paths of integration for which an integral admits a maximum or minimum value. The theory of the calculus of variations was formulated by Euler in the middle part of the 18th century. Originally, the theory was applied to solve dynamic problems in classical mechanics. Today, the theory finds its wide range of applications including control theory, electromagnetic theory, aerodynamics, optics, operational research and the special theory of relativity.

The following illustrations indicate the possibilities of applying the theory of the calculus of variation to deal with some problems in the system effectiveness analysis.

The nature of the optimization problem depends on the given system dynamic equations

$$\dot{\overline{X}} = H\,(\overline{X},\ \overline{M},\ t) \tag{9}$$

If the boundary condition for the control variables are unrestricted, i.e.:

$$-\infty < \overline{M} < \infty \tag{10}$$

then the problem of minimizing or maximizing the integral

$$\overline{E} = \int_{t_o}^{t_f} F(\overline{X},\ \overline{M},\ t)\ dt \tag{11}$$

when the time limits are explicitly known, is referred to as the Lagrange problem. Two other fundamental problems in the calculus of variationsare the Mayer problem and the Bolza problem. The Mayer problem requires us to find $\overline{M}(t)$ such that the function $G(\overline{X},\overline{M},t)$ evaluated at the end points is minimum or maximum, i.e.:

$$\overline{E} = G(\overline{X},\overline{M},t)\Big]_{t_o}^{t_f} \qquad (12)$$

The Bolza problem requires us to find M(t) such that the function:

$$\overline{E} = G(\overline{X},\overline{M},t)\Big]_{t_o}^{t_f} + \int_{t_o}^{t_f} F(\overline{X},\overline{M},t)\ dt \qquad (13)$$

is minimum or maximum. The Bolza problem is the most general problem in the calculus of variations because the function to be optimized contains not only the performance criterion function $F(\overline{X},\overline{M},t)$ but also the constraint function $G(\overline{X},\overline{M},t)$.

In order to demonstrate the strength of Euler's principle, consider the Lagrange problem which requires us to optimize the integral:

$$\overline{E} = \int_{t_o}^{t_f} F(\overline{X},\overline{M},t)\ dt \qquad (14)$$

Referring to equation (2), $\overline{M}$ is a function of $\overline{X}$, $\dot{\overline{X}}$ and t, therefore equation (14) can be written as:

$$\overline{E} = \int_{t_o}^{t_f} F(\overline{X},\dot{\overline{X}},t)\ dt \qquad (15)$$

The necessary condition for minimizing or maximizing equation (15) is that the Euler-Lagrange differential equation [3] must be satisfied; that is:

$$\nabla_{\overline{X}}F - \frac{d}{dt}(\nabla_{\dot{\overline{X}}}F) = 0 \qquad (16)$$

In which $\nabla_{\overline{X}} F$ is defined as $\sum_{i=1}^{n} fi \cdot x_i$

In the case of one-dimensional problem, equation (16) reduces to:

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0$$

In which $f(x, \dot{x}, t)$ is the function to be optimized.

Example 1: <u>A simple case</u>

Consider a simple one dimensional problem whose dynamic equation is given by:

$$\dot{x} = m \tag{18}$$

and the initial conditions are given by:

$$\left.\begin{aligned} x(o) &= X_o \\ \dot{x}(o) &= 0 \end{aligned}\right\} \tag{19}$$

The problem is to find m(t) such that the performance criterion integral:

$$E = \int_{t_o}^{t_f} (x^2 + m^2)\, dt \tag{20}$$

will be minimized.
This is the Lagrange problem with

$$f(x,m,t) = x^2 + m^2 = x^2 + \dot{x}^2 \tag{21}$$

Application of equation (17) results in:

$$\ddot{X} - x = 0 \tag{22}$$

The general solution to equation (22) is:

$$x = Ae^t + Be^{-t} \tag{23}$$

The derivative of equation (23) is:

$$\dot{x} = Ae^t - Be^{-t} \tag{24}$$

If initial conditions are applied to equations (23) and (24), the results are:

$$A = B = \tfrac{1}{2}X_o \tag{25}$$

and equations (23) and (24) become

$$x = X_o \cosh t \tag{26}$$
$$\dot{x} = X_o \sinh t \tag{27}$$

hence $m(t) = X_o \sinh t$ (28)

which is the solution to the problem. In some complicated problems (example 2, below) the application of the calculus of variations usually leads to solving a nonlinear two-point boundary-value problem. In which case, an analytical solution becomes extremely difficult and numerical techniques are usually employed to obtain approximate answers.

Example 2: <u>The general problem</u>

Consider the conditions given in example 1 with the dynamic equation given in the general form.

$$\dot{x} = f_1(x, m, t) \tag{29}$$

Equation (20) becomes

$$e = \int_{t_o}^{t_f} \left[f_2(x,m,t) + \lambda f_1(x,m,t)\right] dt \tag{30}$$

This is known as the isoperimetric problem in which λ is the Lagrange multiplier. Equation (30) can be transformed to a standard Lagrange problem by letting

$$h(x,m,t) = f_2(x,m,t) + \lambda f_1(x,m,t) \tag{31}$$

Equation (17) will therefore become:

$$\frac{\partial h}{\partial x} - \frac{d}{dt}\left(\frac{\partial h}{\partial \dot{x}}\right) = o \tag{32}$$

Equation (32) is usually a nonlinear differential equation. In order to reduce the complexity of the problem, the Lagrange

problem is sometimes transformed to the Mayer problem[4]. Generally, if the end values of x are specified, the Lagrange problem

$$e = \int_{t_o}^{t_f} h(x_1, \dot{x}_1, t)\, dt \tag{33}$$

may be transformed to the Mayer problem by introducing an auxiliary variable x_2 such that

$$\left.\begin{aligned} \dot{x}_2(t_f) &= h(x_1,\ \dot{x}_1,\ t_f) \\ \dot{x}_2(t_o) &= 0 \end{aligned}\right\} \tag{34}$$

The problem then becomes one of minimizing

$$e = \int_{t_o}^{t_f} \dot{x}_2(t_f)\, dt = x_2 \Big]_{t_o}^{t_f}$$

which is indeed the Mayer problem.

Example 3. <u>Application of General Solution</u>

Consider the conditions given in example 2 wherein the dynamic equation is given by:

$$\dot{x}_1 = -x_1 + m \tag{35}$$

Let $f(x,m,t) = x_1^2 + m^2$, the same performance criterion as was given in example 1. Hence, the integral to be optimized is

$$e = \int_{t_o}^{t_f} \left[(x_1^2 + m^2) + \lambda(-x_1 + m)\right] dt \tag{36}$$

By introducing a new coordinate x_2 such that

$$\left.\begin{aligned} \dot{x}_2(t_f) &= (x_1^2 + m^2) + \lambda(-x_1 + m) \\ \dot{x}_2(t_o) &= o \end{aligned}\right\} \tag{37}$$

The problem now becomes one of optimizing

$$e = \int_{t_o}^{t_f} \dot{x}_2(t_f)\, dt = x_2 \Big|_{t_o}^{t_f} \tag{38}$$

This technique of transformation was employed by Pontryagin in his formulation of the maximum and minimum principle which will be discussed in Section IV.

SECTION IV

THEORY OF MAXIMUM AND MINIMUM APPROACH

In the previous discussion, it was stated that one of the conditions for formulating the calculus of variations problem was that the boundary condition for the control variables be unrestricted (i.e. from $-\infty$ to $+\infty$). In reality, control variables are a function of resources, and are therefore always bounded. Since the applications of the calculus of variation are limited to idealized conditions (i.e. equation 10 must be satisfied), a different approach to the optimization problem is required. The theory of maximum and minimum that was introduced by Pontryagin resolves this shortcoming. In essence, the Pontryagin problem is similar to the Mayer problem except that the Pontryagin formulation allows the control variables to be restricted, i.e.:

$$|\bar{M}| \leq \bar{M}_o \tag{39}$$

The motivation behind Pontryagin's formulation is relevant in the present discussion. Recall the isoperimetric problem considered in example 2, in which it was given that

$$\dot{x} = f_1(x,m,t) \tag{40}$$

and

$$e = \int_{t_o}^{t_f} \left[f_2(x,m,t) + \lambda f_1(x,m,t) \right] dt \tag{41}$$

where $f_2(x,m,t)$ is the function to be optimized. If equations (40) and (41) are written in their general vector forms:

$$\dot{\bar{X}} = F\,(\bar{X},\bar{M},t) = \sum_{i=1}^{n} f_i\,(\bar{X},\bar{M},t) \tag{42}$$

and

$$\bar{E} = \int_{t_o}^{t_f} \left[f_{n+1}(\bar{X},\bar{M},t) + \sum_{i=1}^{n} \bar{\lambda}_i f_i(\bar{X},\bar{M},t) \right] dt \tag{43}$$

or

$$\bar{E} = \int_{t_o}^{t_f} \left[f_{n+1}(\bar{X},\bar{M},t) + \sum_{i=1}^{n} \lambda_i f_i(\bar{X},\bar{M},t) \right] dt \tag{44}$$

and the same procedure is followed as was done in transforming the Lagrange problem to the Mayer problem, then:

$$\left.\begin{aligned} \dot{x}_{n+1}(t_f) &= f_{n+1}(\bar{X},\bar{M},t_f) \\ \dot{x}_{n+1}(t_o) &= o \end{aligned}\right\} \tag{45}$$

In the special case when there is no constraint, equation (44) becomes

$$\bar{E} = \int_{t_o}^{t_f} \dot{x}_{n+1}(t_f)\, dt = x_{n+1} \Big]_{t_o}^{t_f} \tag{46}$$

Therefore, the problem of optimizing the integral given by equation (44) becomes the problem of optimizing the (n+1) coordinate, $x_{n+1}(t_f)$, at the terminal point. For example, consider a system whose dynamic equations are given by

$$\dot{x}_i = f_i(\bar{X},\bar{M},t) \qquad i = 1, 2.....n \tag{47}$$

the problem is to find a control vector $\bar{M}$, such that the system changes its state from $\bar{X}(t_o) = \bar{X}_o$ to $\bar{X}(t_f) = \bar{X}_f$ at a minimum time. In classical mechanics, this is immediately recognized as the Brachistochrone problem in which the following integral has to be minimized*:

$$t = \frac{1}{\sqrt{2g}} \int_o^b \frac{\sqrt{1+y'^2}}{\sqrt{y}}\, dx \tag{48}$$

When Pontryagin's principle is applied to the problem, it is only necessary to minimize the integral, $\min\limits_{\bar{M}} \int_{t_o}^{t_f} 1\, dt$, with

*See Appendix I

equation (47) as the constraint. The procedure requires the transformation of the Lagrange problem to the Mayer problem by letting

$$\dot{x}_{n+1}(t_f) = 1 \quad \text{and} \quad \dot{x}_{n+1}(t_o) = o$$

then,

$$\min_{\overline{M}} \int_{t_o}^{t_f} 1 \, dt = \min_{\overline{M}} \int_{t_o}^{t_f} \dot{x}_{n+1}(t_f) \, dt = \min_{\overline{M}} (x_{n+1}) \Big]_{t_o}^{t_f}$$

Since $x_{n+1} = t_f - t_o$, minimization of time means optimization with respect to the new coordinate $x_{n+1}(t_f)$.

In other problems when $\dot{x}_{n+1}(t_f)$ is not equal to a constant, $\dot{x}_{n+1}(t_f)$ may be expanded into a Taylor series:

$$\dot{x}_{n+1} = \frac{\partial x_{n+1}}{\partial x_1} \dot{x}_1 + \frac{\partial x_{n+1}}{\partial x_2} \dot{x}_2 + \ldots\ldots\ldots + \frac{\partial x_{n+1}}{\partial x_n} \dot{x}_n \qquad (49)$$

The standard procedure of transforming the Lagrange problem to the Mayer problem will yield:

$$x_{n+1} = b_1 x_1(t_f) + b_2 x_2(t_f) + \ldots\ldots \; b_n x_n(t_f) \qquad (50)$$

Equation (50) may be written in the closed form

$$x_{n+1}(t_f) = \sum_{i=1}^{n} b_i x_i(t_f) \qquad (51)$$

Equation (51) is referred to as the Pontryagin function. The minimum time example discussed above is the special case of equation (51).

Note that the effectiveness equation:

$$E = \sum_{j=1}^{n} \sum_{i=1}^{n} v_i w_i r_{ij} d_i \qquad \text{(Equation 1)}$$

may be written in the generalized form:

$$E = \sum_{i=1}^{n} f_i(\overline{x}, \overline{m}, t) \qquad \text{(Equation 2)}$$

Since both $\overline{X}$ and $\overline{M}$ are function of time, equation (2) may be further simplified to

$$E = \sum_{i=1}^{n} f_i (t) \tag{52}$$

The effectiveness equation written in the form of equation (52) is indeed similar to the Pontryagin function. For this reason, it is recommended that the study of system effectiveness optimization should follow the Pontryagin procedure which will be discussed next.

The task now is to develop a scheme to optimize equation (52). Observe the similarities between equation (52) and the Hamiltonian [5]:

$$H = \sum_{i=1}^{n} p_i \dot{x}_i$$

In which H is the total energy of the conservative system. Pontryagin has proved[6] that maximizing the Hamiltonian is equivalent to minimizing equation (52) or vice-versa. The formal proof of the Pontryagin principle is tedious and abstract, but intuitively one may visualize that minimizing the derivative of a function is equivalent to locating the maximum value of that function.

Example 4: <u>The Pontryagin Principle</u>

Consider example 3 again in which the dynamic equation was given by:

$$\dot{x}_1 = -x_1 + m \tag{53}$$

The performance criterion was to optimize the integral

$$e = \int_{t_o}^{t_f} (x_1^2 + m^2)\, dt \tag{54}$$

It was shown that the solution may be obtained by using the Lagrange multiplier. The purpose of this example is to illustrate the Pontryagin principle.

Let

$$\left.\begin{aligned} \dot{x}_2(t_f) &= x_1^2 + m^2 \\ \dot{x}_2(t_o) &= o \end{aligned}\right\} \qquad (55)$$

Equation (54) then becomes:

$$e = \int_{t_o}^{t_f} \dot{x}_2(t_f)\, dt = x_2 \Big]_{t_o}^{t_f} \qquad (56)$$

The Lagrange problem in equation (54) has now been reduced to the Mayer problem in equation (56). It is now required to optimize $x_2(t_f)$ with its constraint, namely $x_1(t_f)$.

In other words the Pontryagin function

$$P = x_2(t_f) + \lambda_1 x_1(t_f) \qquad (57)$$

must be optimized. Instead of performing the optimization process on equation (57) directly, one may apply the Pontryagin principle to formulate the Hamiltonian and then optimize the Hamiltonian (Refer to equation 52):

$$H = \dot{x}_2 + p_1 \dot{x}_1 \qquad (58)$$

Substituting equations (53) and (55) into equation (58) yields:

$$H = x_1^2 + m^2 + p_1(-x_1 + m) \qquad (59)$$

Differentiating equation (59) with respect to m:

$$\frac{\partial H}{\partial m} = 2m + p_1 = 0$$

from which the optimum control variable, m, is recognized as:

$$m = -\frac{p_1}{2} \qquad (60)$$

The problem is now to find p_1. The derivative of p_1 is related to H in its canonical form:

$$\dot{p}_1 = -\frac{\partial H}{\partial x_1} = -2x_1 + p_1 \tag{61}$$

Equations (53), (59), (60) and (61) together with the initial and final conditions can now be used to solve for p_1 without any difficulty.

It was mentioned earlier that the theory of maximum and minimum is applicable to problems which have limitations on the control variable m. The next example illustrates this point.

Example 5: <u>Constrained Control Variable</u>

Consider a simple problem wherein the dynamic equation is given by:

$$\dot{x}_1 = -x_1 + m \tag{62}$$

The performance criterion requires optimization of the integral:

$$e = \int_{t_o}^{t_f} x_1^2 dt \tag{63}$$

With the following limitation on m: $|m| \leq M_o$.
Following the same procedure that was outlined in example 4:
Let

$$\dot{x}_2(t_f) = x_1^2 \tag{64}$$

Formulating the Hamiltonian

$$H = x_1^2 + p_1(-x_1 + m) \tag{65}$$

In equation (65), H is a linear function of m, therefore the maximum or minimum value of H depends on the magnitude of m and the sign of p_1. If H is to be minimized,

then the optimum m is a switching function:

$$m = M_o \operatorname{sgn} \left|p_1\right| \tag{66}$$

In which sgn is the abbreviation for sign.

The problem now is to find p_1, knowing that (from Hamilton's canonical form):

$$\dot{p} = -2x_1 + p_1 \tag{67}$$

This problem is best solved by analog simulation. The mechanization of equations (62), (64), (66) and (67) results in the configuration illustrated in Figure 1.

It is of interest to examine a physical interpretation of the solution to the problem stated in example 5 in terms of system effectiveness analysis.

Let:

$m \triangleq$ maintainability whose physical limitation is given by $|m| \leq M_o$ man-hours

$\dot{x}_1 \triangleq$ time derivative of target detection error and is a function of the error and maintainability.

$$\dot{x} = -x_1 + m$$

$x_2 \triangleq \int_{t_o}^{t_f} x_1^2 dt$ is the integral square error. The performance criterion requires this quantity to be minimized.

p_1 is the state variable of the adjoint system, or the control generator which receives feedback information from the system output and generates instantaneous values of m. Hence, the cost of maintaining the system under the stated performance criterion may be calculated by viewing the instantaneous value of m.

The measuring unit shown in Figure 1 establishes the magnitude of the error, x_2. The figure also provides insight into the basic philosophy of simulating a one dimensional problem. The same theory applies, of course, to multi-dimensional problems. Therefore, among other applications, the Pontryagin principle is a powerful tool for system analysts who are dealing with optimization problems.

Although the theory of maximum and mimimum discussed in this section is recognized as both practical and useful, it should not be regarded as the solution to all optimization problems. On the contrary, the application of Pontryagin's principle almost inevitably leads to the solution of a set of nonlinear differential equations which cannot readily be solved by analytical means. In such cases, digital computer techniques must be employed to obtain numerical solutions. Another drawback is that the final state of the system must be known and well defined in order to permit the coordinate transformation. That is, the transformation from the Lagrange problem to the Mayer problem. In many practical problems, the final state of the system is usually not known. Under such conditions, the optimization procedure should be broken down into steps; where each step is an optimization problem by itself. This process is known as dynamic programming and is discussed next.

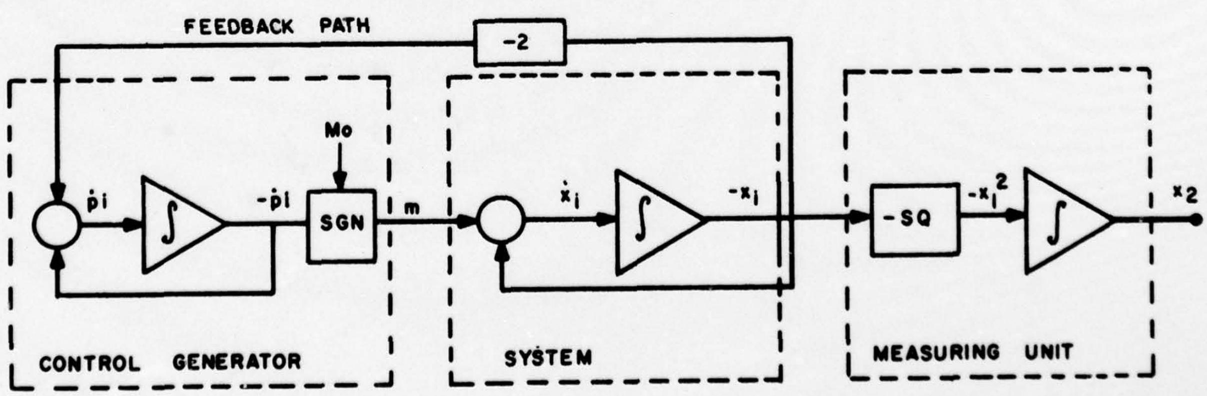

FIGURE 1. ANALOG SIMULATION DIAGRAM

SECTION V

DYNAMIC PROGRAMMING APPROACH

Of the three techniques which were selected for discussion in this paper, perhaps the most powerful one is the principle of dynamic programming which developed by R. Bellman. As mentioned in the closing of the last section, the principle of dynamic programming finds its useful applications in dealing with step by step decision-making problems. In any large system, it is apparent that the derivation of a systematic procedure for obtaining an optimum solution to the problem is extremely difficult. For example, given a difference equation:

$$x(k + 1) = x(k) + m(k) \tag{68}$$

In which x is the state variable, m is the control variable, and it is assumed that $x(o) = X_o$. The problem is to find m(k) such that y in equation (69) is minimum.

$$y \triangleq \sum_{k=0}^{3} x^2(k) \tag{69}$$

Equation (69) may be written as

$$y = \left[x(o)\right]^2 + \left[x(1)\right]^2 + \left[x(3)\right]^2 \tag{70}$$

Substituting equation (68) into equation (70) results in:

$$y = \left[x(o)\right]^2 + \left[x(o) + m(o)\right]^2 + \left[x(1) + m(1)\right]^2 + \left[x(2) + m(2)\right]^2 \tag{71}$$

$$\left.\begin{array}{ll} \text{Since} & x(1) = x(o) + m(o) \\ \text{and} & x(2) = x(1) + m(1) = x(o) + m(o) + m(1) \end{array}\right\} \tag{72}$$

Equation (71) becomes:

$$y = [x(o)]^2 + [x(o) + m(o)]^2 + [x(o) + m(o) + m(1)]^2 + [x(o) + m(o) + m(1) + m(2)]^2 \quad (73)$$

Equation (73) contains four independent variables, namely x(o), m(o), m(1), and m(2). The problem now is to find m(o), m(1), and m(2) such that equation (69) is minimized. According to the theory of differential calculus, the procedure would be as follows:

$$\frac{\partial y}{\partial m(o)} = 6x(o) + 6m(o) + 4m(1) + 2m(2) = \quad (74)$$

$$\frac{\partial y}{\partial m(1)} = 4x(o) + 4m(o) + 4m(1) + 2m(2) = 0 \quad (75)$$

$$\frac{\partial y}{\partial m(2)} = 2x(o) + 2m(o) + 2m(1) + 2m(2) = 0 \quad (76)$$

Equations (74), (75) and (76) are solved by simple matrix algebra. By applying differential calculus techniques, the solution of the problem generally leads to solving a k x k matrix. If k equals 30 instead of 3, the computation is so enormous that it becomes difficult to handle even with digital computers. Therefore, a simplified procedure is required to handle problems of this nature. A new procedure may be derived by applying the fundamental principle of dynamic programming - the Principle of Optimality[7], which states:

> An optimum policy has the property that whatever the initial state and the initial decision, the remaining decision must form an optimal-control strategy with respect to the state resulting from the first decision.

In order to illustrate the principle of optimality, consider the same example discussed above. Rewrite equations (68) and (69):

$$x(k + 1) = x(k) + m(k) \quad (77)$$

$$y = \sum_{k=0}^{3} x^2(k) \tag{78}$$

With the terminal conditions given

$$x(o) = X_o \tag{79}$$

and $$m(3) = 0 \tag{80}$$

Equation (79) specifies the initial state of the system. The subsequent state will depend on X_o. Equation (80) specifies the final control effect is zero. The problem is to find m(o), m(1), and m(2) such that

$$y = \left[x(o)\right]^2 + \left[x(1)\right]^2 + \left[x(2)\right]^2 + \left[x(3)\right]^2 \tag{81}$$

is minimized. The procedure of applying the principle of optimality involves the division of the problem into four steps. Step 1: let $y_o = x(o)^2$, find m(o) such that y_1 is a *minimum*, and represent the optimum y_o by y_o^o. Step 2:

let $$y_1 = y_o^o + \left[x(1)\right]^2 \tag{82}$$

and $$y_1^o = \textit{optimum value of } y_1$$

similarly, step 3: let

$$y_2 = y_1^o + \left[x(2)\right]^2 \tag{83}$$

$$y_2^o = \text{optimum value of } y_2$$

and finally, step 4: let

$$y_3 = y_2^o + \left[x(3)\right]^2 \tag{84}$$

$$y_3^o = \textit{optimum value of } y_3$$

The procedure outlined above seems to be quite straight forward. However, in step 1, x(o) is already a known quantity, and differentiating y_1 with respect to m(o) does not help to

$$y = \sum_{k=0}^{3} x^2(k) \tag{78}$$

With the terminal conditions given

$$x(o) = X_o \tag{79}$$

and

$$m(3) = 0 \tag{80}$$

Equation (79) specifies the initial state of the system. The subsequent state will depend on X_o. Equation (80) specifies the final control effect is zero. The problem is to find m(o), m(1), and m(2) such that

$$y = \left[x(o)\right]^2 + \left[x(1)\right]^2 + \left[x(2)\right]^2 + \left[x(3)\right]^2 \tag{81}$$

is minimized. The procedure of applying the principle of optimality involves the division of the problem into four steps. Step 1: let $y_o = x(o)^2$, find m(o) such that y_1 is a minimum, and represent the optimum y_o by y_o^o . Step 2:

let

$$y_1 = y_o^o + \left[x(1)\right]^2 \tag{82}$$

and

$$y_1^o = \text{optimum value of } y_1$$

similarly, step 3: let

$$y_2 = y_1^o + \left[x(2)\right]^2 \tag{83}$$

$$y_2^o = \text{optimum value of } y_2$$

and finally, step 4: let

$$y_3 = y_2^o + \left[x(3)\right]^2 \tag{84}$$

$$y_3^o = \text{optimum value of } y_3$$

The procedure outlined above seems to be quite straight forward. However, in step 1, x(o) is already a known quantity, and differentiating y_1 with respect to m(o) does not help to

find the optimum value for m(o). On the other hand, m(3) = 0 is given and this would seem to suggest a likely point at which to begin. Therefore, the preceding steps should be reversed and modified as follows:

Let $$y_2 = \left[x(2)\right]^2 + \left[x(3)\right]^2 \tag{85}$$

$$y_1 = y_2^o + \left[x(1)\right]^2 \tag{86}$$

$$y_o = y_1^o + \left[x(o)\right]^2 \tag{87}$$

Note that y_2 includes two terms because it is a function of m(2) only. The optimum value of x(3) is obtained by having m(3) = 0. That is to say, the term $\left[x(3)\right]^2$ is already optimized.

Substituting equation (77) into equation (85):

$$y_2 = \left[x(2)\right]^2 + \left[x(2) + m(2)\right]^2 \tag{88}$$

$$y_2 = \left[x(2)\right]^2 + \left[x(2)\right]^2 + 2x(2)m(2) + \left[m(2)\right]^2 \tag{89}$$

Differentiating equation (89) with respect to m(2) yields:

$$\frac{\partial y_2}{\partial m(2)} = 2x(2) + 2m(2) = 0 \tag{90}$$

and if we let the optimum m(2) be written as $m^o(2)$, then

$$m^o(2) = -x(2) \tag{91}$$

and the optimum y_2 is obtained by substituting equation (91) into equation (89):

$$y_2^o = \left[x(2)\right]^2 \tag{92}$$

Equation (91) indicates that the necessary control signal occurs at k = 2. Equation (92) provides the necessary information for the next optimization step, which involves the use of equation (86). Hence, substituting equation (92) into equation (86):

$$y = \left[x(2)\right]^2 + \left[x(1)\right]^2 \tag{93}$$

Since x(2) = x(1) + m (1), equation (93) becomes:

$$y_1 = 2\left[x(1)\right]^2 + 2x(1)m(1) + \left[m(1)\right]^2 \tag{94}$$

Differentiating y_1 with respect to m(1) results in:

$$\frac{\partial y_1}{\partial m(1)} = 2x(1) + 2m(1) = 0 \tag{95}$$

Let $m^o(1)$ = optimum value of m(1), then

$$m^o(1) = -x(1) \tag{96}$$

The optimum y_1 is obtained by substituting equation (96) into equation (94), hence

$$y_1^o = \left[x(1)\right]^2 \tag{97}$$

By similar procedures, the optimum m(o) and y_o are found to be:

$$m^o(o) = -x(o) \tag{98}$$

$$y_o^o = \left[x(o)\right]^2 \tag{99}$$

The recurrence process may be carried out by a systematic computer program without any difficulty. The ease of digital solution is the main advantage of the dynamic programming technique - especially when dealing with large multi-stage system optimization problems.

In order to generalize the results obtained from the preceding example, consider a discrete system whose dynamic equation is given by:

$$\bar{X}(k+1) = A(k)\bar{X}(k) + \bar{M}(k) \tag{100}$$

In which $\bar{X}$ is the state vector, $\bar{M}$ is the control vector and A is a matrix. The optimization criterion is given by:

$$y = \sum_{k=o}^{N} F\left[\bar{X}(k),\ \bar{M}(k)\right] \tag{101}$$

Assume the initial state and the final control variables are specified as:

$$\bar{X}(o) = \bar{X}_o \tag{102}$$

$$\bar{M}(N) = \bar{M}_N \tag{103}$$

Then the optimization of y over N steps may be written as

$$y_N^o = \underset{m(o),\ m(1),\ m(2)\ldots.m(N)}{\mathrm{Min}} \sum_{k=o}^{N} F\left[\bar{X}(k),\ \bar{M}(k)\right] \tag{104}$$

Equation (104) may be divided into two steps; with a view toward developing a recursion formula.

$$y_N^o = \underset{m(o),\ m(1)\ldots.m(N)}{\mathrm{Min}} \left\{ F\left[\bar{X}(o), \bar{M}(o)\right] + \sum_{k=1}^{N} F\left[\bar{X}(k),\ \bar{M}(k)\right] \right\} \tag{105}$$

Rewrite equation (105):

$$y_N^o = \underset{m(o)}{\mathrm{Min}} \left\{ F\left[\bar{X}(o),\ \bar{M}(o)\right] + \underset{m(1),\ m(2)\ldots.m(N)}{\mathrm{Min}} \sum_{k=1}^{N} F\left[\bar{X}(k),\ \bar{M}(k)\right] \right\} \tag{106}$$

Let $$y_{N-1}^o = \underset{m(1),\ m(2)\ldots.m(N)}{\mathrm{Min}} \sum_{k=1}^{N} F\left[\bar{X}(k),\ \bar{M}(k)\right] \tag{107}$$

Hence equation (106) becomes

$$y_N^o = \min_{m(o)} \left\{ F\left[\overline{X}(o),\ \overline{M}(o)\right] + y_{N-1}^o \right\} \tag{108}$$

Similarly y_{N-1}^o may be divided into two steps

$$y_{N-1}^o = \min_{m(1)} \left\{ F\left[\overline{X}(1),\ \overline{M}(1)\right] + y_{N-2}^o \right\} \tag{109}$$

In which y_{N-2}^o is defined as

$$y_{N-2}^o = \min_{m(2),\ m(3)\ldots.m(N)} \sum_{k=2}^{N} F\left[\overline{X}(k),\ \overline{M}(k)\right] \tag{110}$$

The procedure for obtaining y_{N-3}^o is the same as above.

Hence a recurrence formula based on equations (104) through (110) is developed:

$$y_N^o = \min_{m(o)} \left\{ F\left[\overline{X}(o),\ \overline{M}(o)\right] + \min_{m(1)} \left\{ F\left[\overline{X}(1),\ \overline{M}(1)\right] + \min_{m(2)} \left\{ \ldots y_{N-k} \right\} \right\} \right\} \tag{111}$$

It should be apparent now that the preceding example is a special case of equation (111) in which $F\left[\overline{X}(k),\ \overline{M}(k)\right] = \left[x(k)\right]^2$.

Of course, the dynamic programming technique has the capability to handle much more complex functions such as the quadratic form

$$F\left[\overline{X}(k),\ \overline{M}(k)\right] = \left[\overline{X}(k) - \overline{X}^d\right]^T Q \left[\overline{X}(k) - \overline{X}^d\right] \tag{112}$$

In which $\overline{X}^d$ is the desired state, Q is a square matrix and

$\left[\overline{X}(k) - \overline{X}^d\right]^T$ is the transpose of $\left[X(k) - \overline{X}^d\right]$.

When the principle of dynamic programming applies to continuous systems, equations (100) and (101) may be written as

$$\dot{\overline{X}}(\overline{X}, \overline{M}, t) = A\overline{X} + \overline{M} \tag{113}$$

$$y = \int_{t_o}^{t_f} F(\overline{X}, \overline{M}, t)\,dt \tag{114}$$

In parallel with the procedure used in discrete systems, separate equation (114) into two steps:

$$y = \int_{t_o}^{t_o+\Delta t} F(\overline{X}, \overline{M}, t)\,dt + \int_{t_o+\Delta t}^{t_f} F(\overline{X}, \overline{M}, t)\,dt \tag{115}$$

Define y^o = optimum value of y, such that

$$y^o\left[\overline{X}(t_o),\ t_o\right] = \underset{\overline{M}(t) \in (t_o, t_f)}{\mathrm{Min}} \left\{ \int_{t_o}^{t_o+\Delta t} F(\overline{X}, \overline{M}, t)\,dt + \int_{t_o+\Delta t}^{t_f} F(\overline{X}, \overline{M}, t)\,dt \right\} \tag{116}$$

The notation $\overline{M}(t) \in (t_o, t_f)$ means $M(t)$ belongs to the set $\left\{t_o \leq t \leq t_f\right\}$

Then:

$$y^o\left[\overline{X}(t_o + \Delta t),\ t_o + \Delta t\right] = \underset{M(t) \in (t_o + \Delta t, t_f)}{\mathrm{Min}} \int_{t_o+\Delta t}^{t_f} F(\overline{X}, \overline{M}, t)\,dt \tag{117}$$

and equation (116) may be written as:

$$y^o\left[\bar{X}(t_o),\ t_o\right] = \underset{\bar{M}(t)\ (t_o,\ t_o + \Delta t)}{\mathrm{Min}} \left\{ \int_{t_f}^{t_o+\Delta t} F(\bar{X},\ \bar{M},\ t)dt \right.$$

$$\left. + \underset{\bar{M}(t)\ (t_o + \Delta t,\ t_f)}{\mathrm{Min}} \int_{t_o + \Delta t}^{t_f} F(\bar{X},\ \bar{M},\ t)\ dt \right\} \qquad (118)$$

Substitute equation (117) into equation (118), and $y^o\left[\bar{X}(t_o),\ t_o\right]$ becomes analogous to equation (108).

$$y^o\left[\bar{X}(t_o),\ t_o\right] = \underset{\bar{M}(t)\in(t_o,\ t_o + \Delta t)}{\mathrm{Min}} \left\{ \int_{t_o}^{t_o+\Delta t} F(\bar{X},\ \bar{M},\ t)dt \right.$$

$$\left. + y^o\left[\bar{X}(t_o + \Delta t)\ t_o + \Delta t\right] \right\} \qquad (119)$$

Through the use of a Taylor expansion and approximation, equation (119) can be reduced to the Hamilton-Jacobi equation:*

$$-\frac{\partial y^o}{\partial t_o} = \underset{\bar{M}(t_o)}{\mathrm{Min}} \left\{ F(\bar{X},\ \bar{M},\ t) + \langle \nabla_{\bar{X}}\ y^o,\ \dot{\bar{X}} \rangle \right\} \qquad (120)$$

The symbol < > stands for vector multiplication which is well known in classical mechanics. Current literature refers to equation (120) as the equation of Dynamic Programming. The method of solving equation (120) is best illustrated by doing an example. Consider a continuous system, whose dynamic equation is given by:

$$\dot{x} = -ax + m \qquad (121)$$

*See Appendix II

The performance criterion is given by

$$y = \frac{1}{2} \int_{o}^{T} (x^2 + m^2)dt \tag{122}$$

The problem is to find m(t) such that equation (122) is minimized. From equation (122), the function $F(\overline{X}, \overline{M}, t)$ is identified as:

$$F(\overline{X}, \overline{M}, t) = \frac{1}{2} (x^2 + m^2) \tag{123}$$

While, the term $\langle \nabla_{\overline{X}} y^o, \dot{\overline{X}} \rangle$ in equation (120) is identified as:

$$\langle \nabla_{\overline{X}} y^o, \dot{\overline{X}} \rangle = \frac{\partial y}{\partial x} (-ax + m) \tag{124}$$

Substituting equations (123) and (124) into equation (120) will result in

$$- \frac{\partial y}{\partial t} = \text{Min}\left\{\frac{1}{2} (x^2 + m^2) + \frac{\partial y^o}{\partial x} (-ax + m)\right\} \tag{125}$$

To find the optimum m(t), differentiate equation (125) with respect to m(t)

$$\frac{\partial}{\partial m(t)} \left(- \frac{\partial y}{\partial t}\right) = m + \frac{\partial y}{\partial x} = 0 \tag{126}$$

Hence the optimum m is found to be

$$m^o = - \frac{\partial y^o}{\partial x} \tag{127}$$

Substituting equation (127) into equation (125):

$$- \frac{\partial y^o}{\partial t} = \left\{\frac{1}{2}\left[x^2 + \left(\frac{\partial y^o}{\partial x}\right)^2\right] + \frac{\partial y^o}{\partial x} \left(-ax - \frac{\partial y^o}{\partial x}\right)\right\} \tag{128}$$

Equation (128) is a non-linear second order partial differential equation. It is best solved by the method of separation of variables. Therefore assume a function y^o which contains the non-linear x term multiplied by another function of time:

$$y^o = \frac{1}{2} p(t)\, x^2 \tag{129}$$

Upon substituting equation (129) into equation (128), it becomes:

$$-\frac{1}{2}\frac{dp}{dt}x^2 = \left\{\frac{1}{2}x^2 + \frac{1}{2}p^2(t)x^2 + p(t)x(-ax - p(t)x)\right\} \tag{130}$$

equation (130) may be simplified to:

$$\frac{dp}{dt} = +p^2 + 2ap - 1 \tag{131}$$

Equation (131) may be integrated to give the following result[8]:

$$t = -\frac{2}{\sqrt{4a^2+4}} \tanh^{-1} \frac{2p + 2a}{\sqrt{4a^2+4}} \tag{132}$$

After simplification, equation (132) may be written

$$p(t) = \sqrt{a^2+1}\, \tanh\left(-t\sqrt{a^2+1}\right) - a \tag{133}$$

Substitute equation (133) into (129) to get:

$$y^o = \frac{1}{2}x^2\left\{\sqrt{a^2+1}\, \tanh\left(-t\sqrt{a^2+1}\right) - a\right\} \tag{134}$$

According to equation (127) the optimum m is the negative partial derivative of y^o with respect to x. Hence

$$m^o = -x\left\{\sqrt{a^2+1}\, \tanh\left(-t\sqrt{a^2+1}\right) - a\right\} \tag{135}$$

This example finds its direct applications in the study of system effectiveness by considering x as an error and m as the maintainability which is responsible for that error. The optimum value for m was found in equation (135). The method of obtaining m from the Jacobi-Hamilton equation may be summarized as follows:

Step 1. Identify $F(\overline{X}, \overline{M}, t)$ and $\langle \nabla_{\overline{X}} y^o, \dot{\overline{X}} \rangle$ for the given problem, and formulate the Hamilton-Jacobi equation.

Step 2. Differentiate the Hamilton-Jacobi equation with respect to M, set the derivative equal to zero and find $\overline{M}$ in terms of $\nabla_{\overline{X}} y^o$.

Step 3. Substitute $\overline{M}$ as a function of $\nabla_{\overline{X}} y^o$ into the Hamilton-Jacobi equation. Assume a solution for y^o is composed of two functions, time and x. Solve the equation by the method of the separation of variables.

Step 4. Obtain $\overline{M}^o$ by forming $\nabla_{\overline{X}} y^o$.

The four steps outlined above will permit utilization of a digital computer to provide approximate answers when analytical solutions of problems discussed herein are impractical.

SECTION VI
SUMMARY AND CONCLUSION

The three mathematical techniques discussed are:

1. The Calculus of Variations
2. The Theory of Maximum and Minimum
3. The Principle of Dynamic Programming

The Calculus of Variations require unbounded control variables. This basic requirement limits the applications of the theory in the study of effectiveness. The theory of Maximum and Minimum does not require unbounded control variables, but it does require well defined end-points. This requirement also limits the applications considerably. The Principle of Dynamic Programming is not limited by these requirements, and would seem to provide the most promising technique to solve optimization problems arising from cost-effectiveness studies. The ultimate goal of the current investigation will be the application of these techniques to the optimization of the C/P* design and trade-off analysis. The work will progress in a general fashion so as to allow either major sub-system trade-offs (e.g., comparison of beam-forming technique) or to allow component trade-offs within sub-systems.

where do we go from here.
This is a text book analysis in mathematical form
Implementation of ideas is not really shown

* Conformal/Planar Array Sonar

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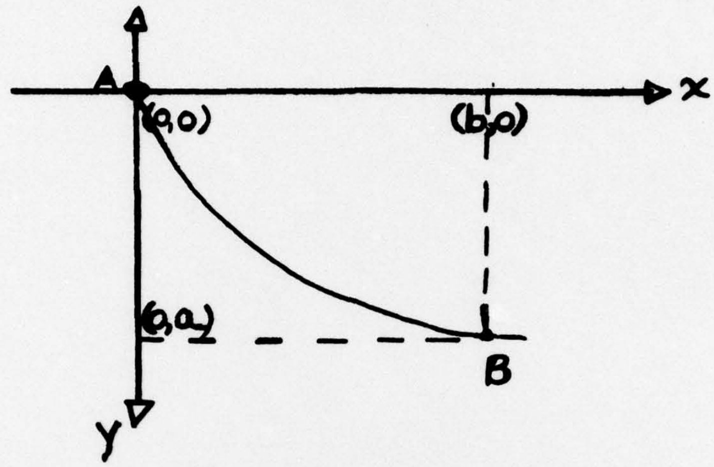
APPENDIX I

THE BRACHISTOCHRONE PROBLEM

Consider an object of mass M as being released from rest with initial velocity equal to zero as shown below:



The problem is to find the path from A to B along which the object may travel in the least amount of time.

The kinetic energy of the system is $\frac{1}{2}Mv^2$ and the potential of the system is Mgy. In which v stands for the velocity of the object and g is the gravitational constant. Assuming that the total energy of the system in conserved, then:

$$\frac{1}{2}Mv^2 = Mgy \tag{1}$$

Simplify equation (1) and let $v = \frac{ds}{dt}$, then

$$\frac{ds}{dt} = 2gy \tag{2}$$

In which

$$ds = \sqrt{(dx)^2 + (dy)^2} \qquad \text{or}$$

$$ds = dx\sqrt{1 + \frac{dy}{dx}^2} \tag{3}$$

Substitute equation (3) into equation (2):

$$dt = \frac{\sqrt{1 + \frac{dy}{dx}^2}}{\sqrt{2gy}}\, dx \tag{4}$$

Integrating equation (4):

$$t = \frac{1}{\sqrt{2g}} \int_o^b \frac{1 + \dot{y}^2}{\sqrt{y}}\, dx \tag{5}$$

In which $\dot{y} = \frac{dy}{dx}$

The solution to equation (5) is, of course, the well known cycloid. (See reference 3).

APPENDIX II

DERIVATION OF THE HAMILTON-JACOBI EQUATION

Rewrite equation (119):

$$y^o\left[\overline{X}(t_o),\ t_o\right] = \underset{\overline{M}(t)\in(t_o,\ t_o+\Delta t)}{\mathrm{Min}} \left\{ \int_{t_o}^{t_o+\Delta t} F(\overline{X},\ \overline{M},\ t)\ dt + y^o\left[\overline{X}(t_o+\Delta t),\ t_o+\Delta t\right] \right\} \quad (1)$$

Note that the integral $\int_{t_o}^{t_o+\Delta t} F(\overline{X},\ \overline{M}\ \ t)dt$ may be expanded into the Taylor series:

$$\int_{t_o}^{t_o+\Delta t} F(\overline{X},\ \overline{M}, t)dt = Z(t_o+\Delta t) = Z(t_o) + \frac{dZ}{dt_o}\Delta t + O(\Delta t^2)$$

Since $Z(t_o) = 0$ and $\frac{dZ}{dt_o} = F(\overline{X},\ \overline{M},\ t_o)$, then equation (2) becomes:

$$\int_{t_o}^{t_o+\Delta t} F(\overline{X},\ \overline{M},\ t)dt = F(\overline{X},\ \overline{M},\ t_o)\Delta t + O(\Delta t^2) \quad (3)$$

The next term in the recursion $y^o\left[\bar{X}(t_o + \Delta t), t_o + \Delta t\right]$ may also be expanded into the Taylor series:

$$y^o\left[\overline{X}(t_o + \Delta t), t_o + \Delta t\right] = y^o\left[\overline{X}(t_o), t_o\right] + \Delta t\left[\frac{\partial y^o}{\partial x_1}\right]\cdot\left[\dot{x}_1\right]$$

$$+ \Delta t\left[\frac{\partial y^o}{\partial x_2}\right]\cdot \dot{x}_2 + \ldots + \Delta t\left[\frac{\partial y^o}{\partial x_n}\right]\cdot \dot{x}_n$$

$$+ \Delta t \frac{\partial y^o}{\partial t_o} + 0(\Delta t^2) \tag{4}$$

Equation (4) may be written as

$$y^o\left[\overline{X}(t_o + \Delta t), t_o + \Delta t\right] = y^o\left[\overline{X}(t_o), t_o\right] + \Delta t\left\{\sum_{i=1}^{n} \frac{\partial y^o}{\partial x_i} \cdot \dot{x}_i\right\}$$

$$+ \Delta t \frac{\partial y^o}{\partial t_o} + 0(\Delta t^2) \tag{5}$$

Define $$\nabla_{\overline{X}}\, y^o = \sum_{i=1}^{n} \frac{\partial y^o}{\partial x_i} \cdot \dot{x}_1$$

Then Equation (5) becomes

$$y^o\left[\overline{X}(t_o + \Delta t), t_o + \Delta t\right] = y^o\left[\overline{X}(t_o), t_o\right]$$

$$+ \Delta t\left\{\left(\nabla_{\overline{X}} y^o\right) \cdot \left(\dot{\overline{X}}\right)\right\} + \frac{\partial y^o}{\partial t_o}\Delta t + 0(\Delta t^2) \tag{7}$$

Substitute equations (3) and (7) into equation (1):

$$y^o\left[\overline{X}(t_o),\ t_o\right] = \underset{\overline{M}(t)\in(t_o,\ t_o+\Delta t)}{\mathrm{Min}} \Big\{ F(\overline{X},\ \overline{M},\ t_o)\Delta t + y^o\left[\overline{X}(t_o),\ t_o\right]$$

$$+ \Delta t\left[\left(\nabla_{\overline{X}} y^o\right)\left(\dot{\overline{X}}\right)\right] + \frac{\partial y^o}{\partial t_o}\Delta t + 0(\Delta t^2)\Big\} \tag{8}$$

Define $(\nabla_{\overline{X}}\, y^o)(\dot{\overline{x}}) = \langle \nabla_{\overline{X}}\, y^o,\ \dot{\overline{X}} \rangle$ and drop terms second order in Δt.

Then equation (8) may be written as

$$-\frac{\partial y^o}{\partial t_o}\,\Delta t = F(\overline{X},\ \overline{M},\ t_o)\Delta t + \langle \nabla_{\overline{X}}\, y^o,\ \dot{\overline{X}} \rangle\,\Delta t \tag{9}$$

Divide equation (9) by Δt:

$$-\frac{\partial y^o}{\partial t_o} = F(\overline{X},\ \overline{M},\ t_o) + \langle \nabla_{\overline{X}}\, y^o,\ \dot{\overline{X}} \rangle \tag{10}$$

Equation (10) is identified as the Hamilton-Jacobi equation.